1 Lagrange Multiplier Test for State Effects

Our model (in equation (1)) is

$$y^*_{nijt} = X_{nit}\beta_j + Z_{nijt}\gamma + \alpha_j y_{nijt-1} + \omega_{nijt} + \varepsilon_{nijt}.$$

In section 5.2, we consider a more general model,

$$y^*_{nijt} = X_{nit}\beta_j + Z_{nijt}\gamma + \alpha_j y_{nijt-1} + \sum_s d_{nis}\tau_{is} + \omega_{nijt} + \varepsilon_{nijt},$$

and we want to test

$$H_0 : \tau_{is} = 0 \;\forall s$$

against the general alternative. The complication is that there are a lot of singularities associated with state dummies d_{nis} and any variables in X_{nit} that are constant over time; i.e., all of the policy variables.

Instead of trying to carefully analytically determine all of the restrictions, we can achieve the same result more generally. Consider a problem,

$$\begin{array}{rcl} \underset{K\times 1}{u} & \sim & N(0,\Omega) \\ \underset{R\times K}{A}\, u & = & 0. \end{array}$$

We can think of u as the vector of K score statistics from a Lagrange Multiplier test and A as the matrix of R restrictions imposed on u associated with state dummies and time-constant elements of X_{nit}. We can decompose Ω as

$$\Omega = C\lambda C'$$

where C is the matrix of eigenvectors of Ω and λ is a diagonal matrix with the nonnegative eigenvalues on the diagonal. The fact that there are R restrictions imposed on u implies that R of the eigenvalues in λ are zero, so we can write

$$\lambda = \begin{pmatrix} \underset{R\times R}{0} & \underset{R\times(K-R)}{0'} \\ \underset{(K-R)\times R}{0} & \underset{(K-R)\times(K-R)}{D} \end{pmatrix}$$

and

$$C = \begin{pmatrix} \underset{R\times R}{C_{11}} & \underset{R\times(K-R)}{C_{12}} \\ \underset{(K-R)\times R}{C_{21}} & \underset{(K-R)\times(K-R)}{C_{22}} \end{pmatrix} = \begin{pmatrix} \underset{R\times K}{C_1} \\ \underset{(K-R)\times K}{C_2} \end{pmatrix}$$

(where C_2 is the matrix of eigenvectors associated with the positive eigenvalues). We can ignore $C_1 u$ because the restrictions imply that it is zero.

Thus, consider

$$C_2 u \sim N\left(0, C_2\Omega C_2'\right).$$

Note that

$$C_2\Omega C_2' = C_2 C\lambda C' C_2'$$

$$
\begin{aligned}
&= C_2 \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} \begin{pmatrix} 0 & 0' \\ 0 & D \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}' C_2' \\
&= \begin{pmatrix} C_{21} & C_{22} \end{pmatrix} \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \begin{pmatrix} 0 & 0' \\ 0 & D \end{pmatrix} \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}' \begin{pmatrix} C_{21}' \\ C_{22}' \end{pmatrix} \\
&= \begin{pmatrix} C_{21} & C_{22} \end{pmatrix} \begin{pmatrix} C_{12}DC_{12}' & C_{12}DC_{22}' \\ C_{22}DC_{12}' & C_{22}DC_{22}' \end{pmatrix} \begin{pmatrix} C_{21}' \\ C_{22}' \end{pmatrix} \\
&= C_{21}C_{12}DC_{12}'C_{21}' + C_{22}C_{22}DC_{12}'C_{21}' + C_{21}C_{12}DC_{22}'C_{22}' + C_{22}C_{22}DC_{22}'C_{22}' \\
&= C_{21}C_{12}D\left(C_{12}'C_{21}' + C_{22}'C_{22}'\right) + C_{22}C_{22}D\left(C_{12}'C_{21}' + C_{22}'C_{22}'\right) \\
&= \left(C_{21}C_{12} + C_{22}C_{22}\right) D \left(C_{12}'C_{21}' + C_{22}'C_{22}'\right) = D.
\end{aligned}
$$

Then, under H_0,

$$D^{-1/2}C_2 u \sim N\left(0, I_{K-R}\right),$$

and

$$\left(D^{-1/2}C_2 u\right)' \left(D^{-1/2}C_2 u\right) \sim \chi^2_{K-R}.$$

Note that, instead of having to analytically determine all of the singularities in u, we need only count the number of positive eigenvalues.[1]

For our problem, the sample test statistic is 1850.7, and it is distributed χ^2_{221} under H_0. This is statistically significant at any relevant size.

[1] There is a roundoff error problem in that some zero eigenvalues will appear to be very small numbers. We use a rule of thumb that any eigenvalue less than 0.0001 is really zero.